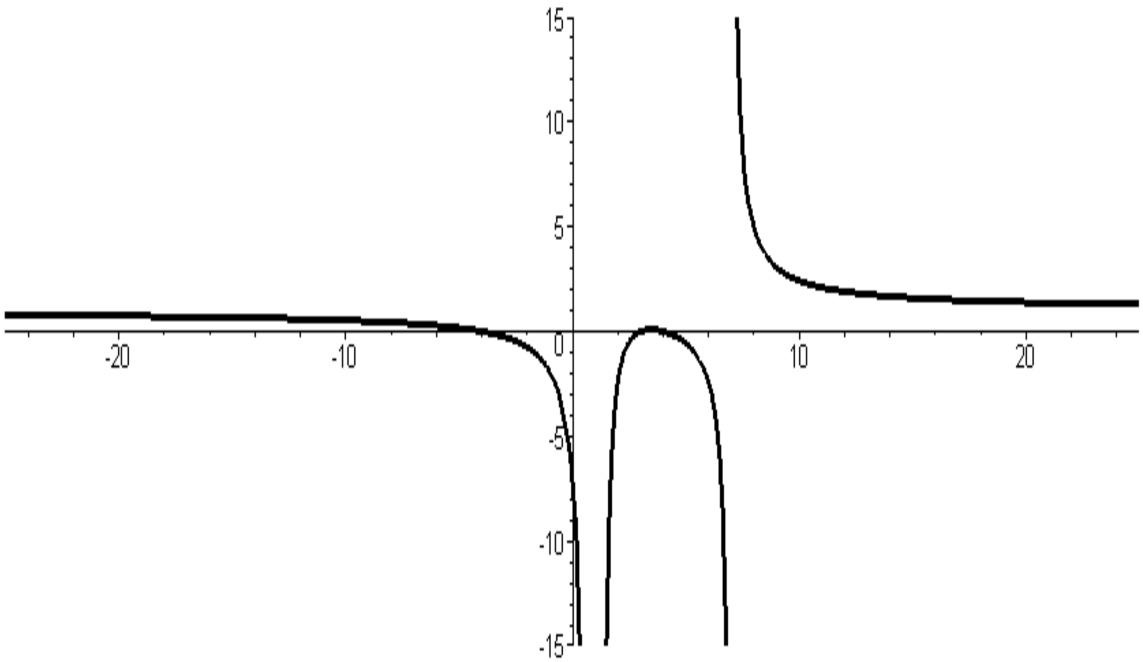
Kaushlesh Biyani

Calculus I

Graphs can be fun!

How many times have we stared a horrible looking function and imagined how its graph would look like. Well, our first inclination would be to take our graphical calculator or ask Maple to graph it. But wait, we have another way to graph that horrible function. A method, unlike our graphical calculator and Maple, which can be used on our Math Test too! Here I would like to introduce our savior, **Calculus**, defined by the Britannica encyclopedia as, a branch of mathematics concerned with the study of such concepts as the rate of change of one variable quantity with respect to another, the slope of a curve at a prescribed point, the computation of the maximum and minimum values of functions, and the calculation of the area bounded by curves. Let us consider this function, $f(x) = \frac{x^3 - 3x^2 - 16x + 48}{x^3 - 9x^2 + 15x - 7}$. When graphed, it looks like the curve below.

Now let us ponder over, "Why the graph of the function looks the way it does?"



The given function is a Rational Function. It is undefined at the points x = 1 and x = 7, as at these points the denominator of the function, $x^3 - 9x^2 + 15x - 7$ equals zero.

This is illustrated on the curve, as the graph is not continuous or defined at these points. Therefore the **domain** of the function is all real numbers except 1 and 7.

The ***x* - intercepts** are the points where the ***y*** co-ordinate of the point on the curve is equal to **0**. This occurs at x = -4, 3 and 4. Thus the curve intercepts the ***x* - axis** at these three points.

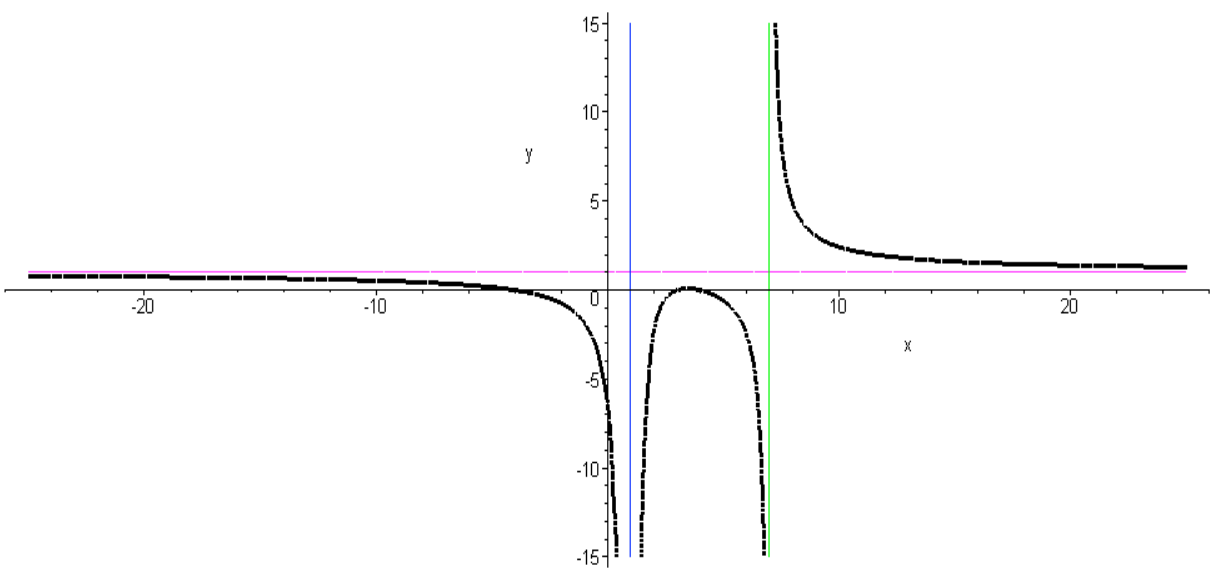
To find the **horizontal asymptote** we need to evaluate

$$\lim_{x \to (-\infty)} \frac{x^3 - 3x^2 - 16x + 48}{x^3 - 9x^2 + 15x - 7}$$

. On the graph, we observe that as x approaches larger and larger values on both sides of the Origin, the value of *f (x)* approaches 1. Therefore the line ***y* = 1** is a horizontal asymptote a horizontal line can be drawn at these point without touching the graph of *f (x)*.

The **vertical asymptotes** of a function occur at the points that lie outside the Domain of the function. The points x = 1, 7 are not in the domain of *f (x)*. Therefore, the lines ***x* = 1** and ***x* = 7** are the vertical asymptotes of *f (x)*. Thus we observe that *f (x)* is not defined for the values x =1, 7 and a vertical line can be drawn at both these points without touching the graph of f *(x)*.

All these lines, x =1, x =7 and y =1, more to the point, the graph of *f (x)* approaches these lines without actually touching them.

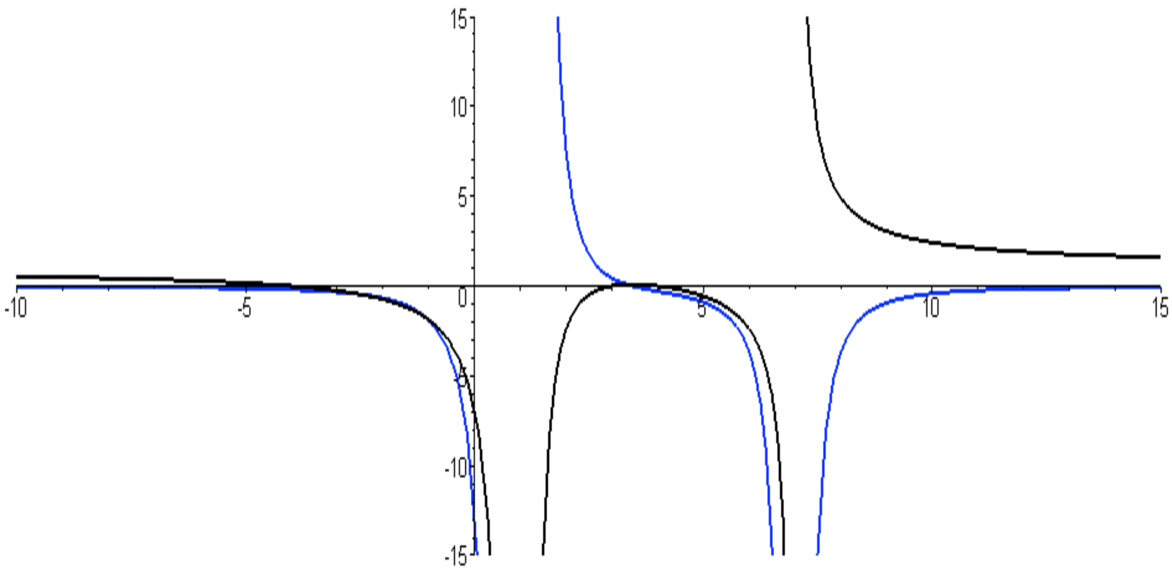


To find the **critical points** of *f (x)*, we set *f '(x)* = 0 and solve for

$$f'(x) = -\frac{2\,(3\,x^3 - 28\,x^2 + 149\,x - 304)}{x^5 - 17\,x^4 + 94\,x^3 - 190\,x^2 + 161\,x - 49} = 0$$

$$2\,(3\,x^3 - 28\,x^2 + 149\,x - 304) = 0$$

On solving we find that *f '(x)* is equal to 0, when x = 3.45046 . We also find that, ***f*** **(** 3.45046 **)** = .86531 . Precisely, a critical point is a zero of *f'(x)*; such a point may or may not be a local extremum – that's what the First Derivative Rule is for. By plotting the graph of the function *f (x)* and its first derivative *f '(x)*, we observe that *f (x)* has a **local maximum** when x = 3.45046 **.** The **local minimum** of *f (x)* **does not exit**



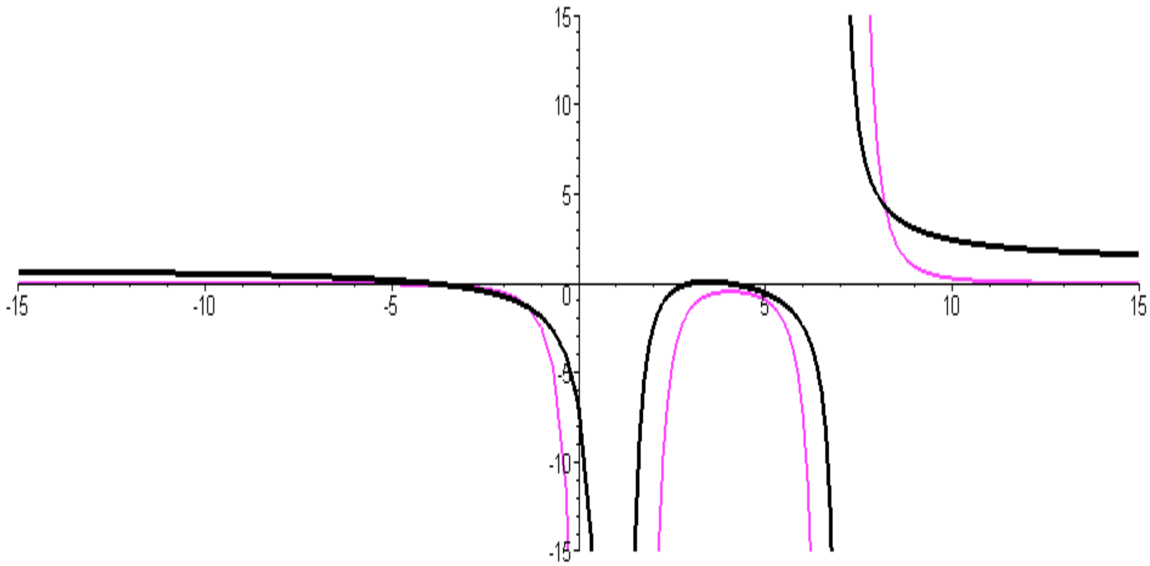
The Local Maximum value of *f* (3.45046) = .86531 is not a Global Maximum as it is evident from the graph of *f (x)* that there exist many values of x for which *f (x)* is greater than .86531 . Therefore the **global maximum and minimum** of *f (x)* **do not exist**

We find that ***f '(x)* > 0** on (1, 3.45046) and **decreasing** otherwise. Thus by the First Derivative Test, *f (x)* is **increasing** on (1, 3.45046). We might be tempted to conclude that the function is decreasing on (- ∞ , 1) and (3.45046 , ∞), but this is not true as the function *f(x)* is discontinuous on x=7. Therefore *f(x)* is **decreasing** on the intervals (- ∞ , 1), (3.45046 , 7) and (7, ∞).

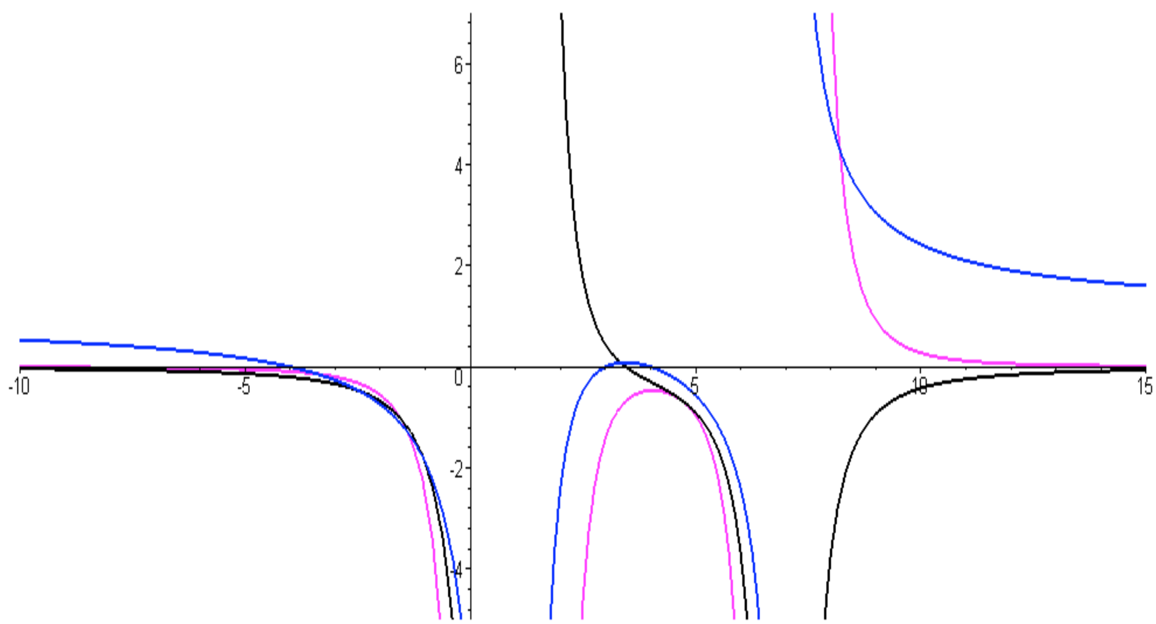
To find the **inflection points** of *f (x)* we equate *f''(x)* = 0, and solve for *x*

$$f''(x) = \frac{6\,(2\,x^4 - 27\,x^3 + 243\,x^2 - 1121\,x + 1983)}{x^7 - 25\,x^6 + 237\,x^5 - 1061\,x^4 + 2339\,x^3 - 2667\,x^2 + 1519\,x - 343} = 0$$

But on solving *f''(x)*, we observe that there are no real values for which *f''(x)* = 0. Therefore the graph of *f (x)* has **no inflection points**. We find that f''***(x)* > 0** on (7, ∞) and ***f''(x)* < 0** on (- ∞ , 7).



By the Second Derivative Test, ***f (x)*** is **concave upward** on (7, ∞) **concave downward** on (- ∞ , 7) .



In the analysis of the given function, we found out a few important things –

- The **domain** of the function was found out to be all real numbers except 1 and 7.
- The ***x*-intercept** of the curve is at $y = 0$ and the ***y*-intercept**s are at $x = -4$, 3 and 4
- The **horizontal asymptote** of the function is at $y = 1$ and the **vertical asymptotes** are at $x = 1$ and 7.
- The function is **increasing** on the interval (1, 3.45046) and **decreasing** on the intervals $(-\infty, 1)$, $(3.45046, 7)$ and $(7, \infty)$.
- The **local maximum** is at $x = 3.45046$ and it does not have a **local minimum**.
- The **global maximum and minimum** of the function do not exist.
- The graph of the function is **concave upward** on $(7, \infty)$; **concave downward** on $(-\infty, 1)$ and $(1, 7)$ and does not have an **inflection point**.

All the above values are specific to the given function $f(x) = \dfrac{x^3 - 3x^2 - 16x + 48}{x^3 - 9x^2 + 15x - 7}$.

But if we find all the above values, namely, the domain of the function, its x and y intercepts, its horizontal and vertical asymptotes, its intervals on which the function increases or decreases and maximum or minimum values (with the help of the function's first derivative and the first derivative test) and the intervals of concavity upward / downward and the inflection points (with the help of the function's second derivative and the second derivative test) for any given function, then we will be able to graph it accurately. This might seem to be a bit difficult and too time consuming to some of you. But, trust me guys, this is a much better way of learning graphs and their behavior than just asking your graphical calculator or maple to do it for you. And most importantly you will be able to use the above method on your math test, unlike maple or your graphical calculator which hardly any Calc I professor would allow to be used on tests.

So what are you waiting for! Go-ahead, grab the ugliest function you have ever know and graph it to oblivion, with the easy method just explained to you!